ON THE SYNTHESIS OF LINEAR AUTOMATIC CONTROL SYSTEMS WITH BOUNDS ON THE DISPLACEMENTS OF THE REGULATING ELEMENTS USING THE CRITERION OF MINIMUM MEAN SQUARE ERROR AT ANY GIVEN INSTANT OF TIME

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1. For the sake of greater compactness we will make use of the matrix notation. The number of rows and columns of the matrices will, as a rule, not be mentioned. Unless otherwise specified, they can be arbitrary as long as the products occurring in formulas have a meaning. The unit matrix will be denoted by the letter E, or E_n when it is necessary to give its order. The transposed matrix of some matrix A we shall denote by A'; thus, in particular, x' will be a row if x is a column. If A is a square matrix then $|A|$ will represent its determinant; sp (A) will denote its trace (that is, the sum of the elements of the main diagonal). If $A = (a_{ij})$, $B = (b_{ij})$ are matrices with the same number of rows and the same number of columns then the following relations hold:

$$\mathrm{sp}\,(AB') = \mathrm{sp}\,(BA') = \mathrm{sp}\,(A'B) = \mathrm{sp}\,(B'A) = \sum_{i,\,j} a_{ij} b_{ij} \qquad (1.1)$$

Besides the ordinary matrices whose elements are numbers or functions of time, we will deal with columns consisting of random quantities or random processes (which, obviously, in particular cases can coincide with definite numbers or functions of time). The arguments of the functions and of the random functions of time will always be indicated.

For our purposes the random quantities will be completely characterized by their initial second moments. If $a(t)$ and $b(t)$ are random column vectors, then $\psi_{ab}(t, \tau)$ will denote the matrix of the second moments of their components

$$\psi_{ab}(t, \tau) = M[a(t) b'(\tau)] \tag{1.2}$$

Here M stands for the mathematical expectation.

It is obvious that

$$\psi_{ab}(t, \tau) = \psi_{ba}'(\tau, t) \tag{1.3}$$

In place of $\psi_{ab}(t, t)$ we shall write $\psi_{ab}(t)$. We will assume the existence of finite second moments, except for the case of white noise when the matrix of the second moments has the form

$$\psi_{aa}(t, \tau) = A(t)\ \delta(t - \tau) \tag{1.4}$$

2. Suppose that we are given a dynamic system described by the equation

$$x(t) = s(t) - \int_0^t h(t, \tau) y(\tau)\, d\tau \tag{2.1}$$

Here $x(t)$ is a column of the coordinates of the system; $s(t)$ is a column of given random quantities which characterize the external reactions and state of the system at the initial instant of time $t = 0$; $y(\tau)$ is a column of the coordinates of the regulating elements, and $h(t, \tau)$ is a given matrix.

The regulating reactions $y(t)$ are formed from the given column vector ρ and from the values of the given random column vector $r(\tau)$, $0 \leqslant \tau \leqslant t$, as a sum of two linear operators on ρ and $r(\tau)$. The set of all random column vectors, obtained in this manner for all possible linear operators which are commutative with the operation of mathematical expectation, we will denote by $L(t)$. $L(t)$ is exactly the set of all random column vectors a for which the function $\psi_{\beta a} = 0$, if $\psi_{\beta\rho} = 0$ and $\psi_{\beta r}(\tau) = 0$ $(0 \leqslant \tau \leqslant t)$.

There arises the problem of finding a $y(t) \in L(t)$ $(0 \leqslant t \leqslant T)$ for which the mean square deviation of the system from a given position x_T at a given instant of time $T > 0$ will be minimal. Without restricting the generality, one may assume that $x_T = 0$, and

$$\varepsilon^2(T) = M[x'(T) x(T)] = \mathrm{sp}\ \psi_{xx}(T) = \min \tag{2.2}$$

In order to achieve uniqueness for the optimal system, we impose upon $y(t)$ restrictions of the form

$$l_i \leqslant k_i \qquad (i = 1, \ldots, m) \tag{2.3}$$

where the k_i are given numbers, and the l_i are functionals of the form

$$l_i = M \int_0^T \lambda_i(t)\, y_i^2(t)\, dt \qquad (i = 1, \ldots, m) \tag{2.4}$$

that is, the l_i are the values of the integral of the mean square values of the coordinates of the ith regulating element; the $\lambda_i(t)$ are given weight functions. They can be arbitrary functions of time which for all t are greater than some positive number

$$\lambda_i(t) \geqslant \lambda > 0 \qquad (i = 1, \ldots, m) \tag{2.5}$$

In the following section there will be given a solution of this general problem. The results are applied to a problem on the synthesis of an automatic control system.

3. Let us assume that we have found the optimal system. Then

$$\delta\varepsilon^2(T) \geqslant 0 \tag{3.1}$$

for all variations $\delta y(t)$ that satisfy the conditions

$$l_i + \delta l_i \leqslant k_i \qquad (i = 1, \ldots, m) \tag{3.2}$$

(the l_i are defined by the Equation (2.4)).

As is usually done, we assume that the variations are infinitesimally small. The inequalities (3.2) will therefore be valid for arbitrary variations if $l_i < k_i$. For those values of the index i for which $l_i = k_i$, the inequalities (3.2) are equivalent to the inequalities

$$\delta l_i \leqslant 0 \tag{3.3}$$

Thus, we have the inequalities (3.1) for all variations that satisfy the inequalities (3.3) for those i for which $l_i = k_i$. Since this is true, in particular, when $\delta l_i = 0$, there exist Lagrange multiples μ_i such that

$$\delta\varepsilon^2(T) + \sum_i \mu_i\, \delta l_i = 0 \tag{3.4}$$

for arbitrary variations $\delta y(t)$. The summation is performed here over these values of i for which $l_i = k_i$. But by assuming that $\mu_i = 0$ for the other values of i (for which $l_i < k_i$) one can sum over all the indices (from 1 to m). If one varies only the ith component of the column $y(t)$, then $\delta l_i \neq 0$, while $\delta l_j = 0$ $(i \neq j)$, and by (3.4), (3.3) and (3.1), $\mu_i > 0$. Thus we have the condition

$$\mu_i \geqslant 0, \quad \mu_i = 0, \qquad \text{if} \quad l_i < k_i \ (i = 1, \ldots, m) \tag{3.5}$$

If we set

$$\beta(t) = \left\| \begin{matrix} \mu_1\lambda_1(t) & \ldots & 0 \\ \ldots & \ldots & \ldots \\ 0 & \ldots & \mu_m\lambda_m(t) \end{matrix} \right\| \tag{3.6}$$

and take account of the relations (2.2) and (2.4), we may write the Equations (3.4) in the form

$$\mathrm{sp}\,[\delta\psi_{xx}(T) + \delta \int_0^T \beta(t)\,\psi_{yy}(t)\,dt] = 0 \tag{3.7}$$

From this we obtain directly

$$\mathrm{sp}\,[\psi_x\delta_x(T) + \int_0^T \beta(t)\,\psi_y\,\delta_y(t)\,dt] = 0 \tag{3.8}$$

By Equation (2.1) we have

$$\delta x(T) = -\int_0^T h(T, t)\,\delta y(t)\,dt \tag{3.9}$$

Let us substitute this expression into the Equation (3.8), and apply Formula (1.1). Then we obtain

$$\mathrm{sp} \int_0^T M[w(t)\,\delta y'(t)]\,dt = 0 \tag{3.10}$$

Here $w(t)$ is defined by the formula

$$w(t) = \beta(t)\,y(t) - h'(T, t)\,x(T) \tag{3.11}$$

Since $y(t) \in L(t)$, this is equivalent to the equations

$$\psi_{wp}(t) = 0, \qquad \psi_{wr}(t, \tau) = 0 \qquad (t \geqslant \tau) \tag{3.12}$$

One can prove, conversely, that every control system that satisfies the conditions (3.12) and (3.15) minimizes $\epsilon^2(T)$ among all systems that satisfy the restrictions (2.3). For this purpose let us assume that $y_*(t)$ is the response of such a system. Then we set

$$\begin{aligned} \varepsilon^2(T) &= M[x'(T)\,x(T)], \qquad \eta^2(T) = \mathrm{sp}\int_0^T \beta(t)\,\psi_{yy}(t)\,dt \\ \varepsilon_*^2(T) &= M[x_*'(T)\,x_*(T)], \qquad \eta_*^2(T) = \mathrm{sp}\int_0^T \beta(t)\,\psi_{y_*y_*}(t)\,dt \end{aligned} \tag{3.13}$$

(In the definition of $\eta_*^2(T)$, $\beta(t)$ has the same meaning as in $\eta^2(T)$!). Now we define the differences

$$\Delta x(t) = x_*(t) - x(t), \qquad \Delta y(t) = y_*(t) - y(t) \tag{3.14}$$

and $\Delta\epsilon^2(T)$, $\Delta\eta^2(T)$ in an analogous manner. From the Equations (3.12) it follows that the symbol (3.10) can be replaced by the symbol Δ. Since this is also the case for the Equation (3.9), we obtain an equation of finite differences

$$\mathrm{sp}\,\{\psi_{x\Delta x}(T) + \int_0^T \beta(t)\,\psi_{y\Delta y}(t)\,dt\} = 0 \tag{3.15}$$

This equation corresponds to the equation of variations (3.8). On the basis of the Equations (3.13) we have the relations

$$\Delta\varepsilon^2(T) = 2\,\mathrm{sp}\,\psi_{x\Delta x}(T) + \mathrm{sp}\,\psi_{\Delta x\Delta x}(T) \tag{3.16}$$

$$\Delta\eta^2(T) = 2\,\mathrm{sp}\int_0^T \beta(t)\,\psi_{y\Delta y}(t)\,dt + \mathrm{sp}\int_0^T \beta(t)\,\psi_{\Delta y\Delta y}(t)\,dt \tag{3.17}$$

The condition (3.5) shows that $\eta_*^2(T) \leqslant \eta^2(T)$, i.e. $\Delta\eta^2(T) \leqslant 0$. Since $\beta(t)$ is a positive semidefinite matrix, the second terms on the right-hand sides of Equations (3.16) and (3.17) are nonnegative. Hence,

$$\mathrm{sp}\int_0^T \beta(t)\,\psi_{y\Delta y}(t)\,dt \leqslant 0$$

Then, in view of Equation (3.15), we have

$$\mathrm{sp}\,\psi_{x\Delta x}(T) \geqslant 0$$

From Equation (3.16) it now follows that $\Delta\epsilon^2(T) \geqslant 0$, which was to be proven.

If all the $\mu_i > 0$, i.e. if $\beta(t)$ is a positive definite matrix, we have the strong inequality $\Delta\epsilon^2(T) > 0$. In the case that all $\mu_i > 0$, and, hence, all $l_i = k_i$, the behavior of the optimal system $y(t)$ is definite with one sign.

Thus we have proved that the set of optimal systems corresponds to the set of solutions of the Equations (3.12) with the numbers μ_i satisfying the conditions (3.5).

4. We shall solve the Equations (3.12) with the best mean square approximation $\sigma(r)$ of the random vector $s(T)$ by means of random vectors from $L(r)$. Here $\sigma(r)$ is determined uniquely by the equations

$$\psi_{\sigma\rho}(\tau) = \sigma_{s\rho}(T), \qquad \psi_{\sigma r}(\tau, \vartheta) = \psi_{sr}(T, \vartheta) \qquad (\tau \geqslant \vartheta) \tag{4.1}$$

Let us introduce the auxiliary quantities $\xi(r)$, $\eta(t, r)$ defined in a similar way as the best mean square approximations of the vectors $x(T)$, $y(t)$, respectively, i.e. $\xi(r)$, $\eta(r) \in L(r)$, and

$$\begin{aligned} &\psi_{\xi\rho}(\tau) = \psi_{x\rho}(T), && \psi_{\xi r}(\tau, \vartheta) = \psi_{xr}(T, \vartheta) \\ &\psi_{\eta\rho}(t, \tau) = \psi_{y\rho}(t), && \psi_{\eta r}(t; \tau, \vartheta) = \psi_{yr}(t, \vartheta) \end{aligned} \quad (\tau \geqslant \vartheta) \tag{4.2}$$

In view of the uniqueness of this type of approximations, the Equations (3.11) and (3.12) yield

$$\beta(t)\, \eta(t, \tau) - h'(T, t)\, \xi(\tau) = 0 \qquad (t \geqslant \tau) \tag{4.3}$$

The Equation (2.1) leads to

$$\xi(\tau) = \sigma(\tau) - \int_0^T h(T, t)\, \eta(t, \tau)\, dt \tag{4.4}$$

Obviously, we have

$$\eta(t, \tau) = \eta(t, t) = y(t) \qquad (t \leqslant \tau) \tag{4.5}$$

and (4.4) can, therefore, be written in the form

$$\xi(\tau) = \sigma(\tau) - \int_0^\tau h(T, t)\, \eta(t, t)\, dt - \int_\tau^T h(T, t)\, \eta(t, \tau)\, dt \tag{4.6}$$

Suppose that $|\beta(t)| \neq 0$, i.e. all $\mu_i > 0$. In this case we obtain with the aid of (4.3)

$$\sigma(\tau) = \gamma(\tau)\, \xi(\tau) - \int_0^\tau \dot{\gamma}(t)\, \xi(t)\, dt \tag{4.7}$$

where

$$\gamma(\tau) = E + \int_\tau^T h(T, t)\, \beta^{-1}(t)\, h'(T, t)\, dt \tag{4.8}$$

Since $\gamma(\tau)$ is a positive definite matrix, $|\gamma(\tau)| \neq 0$. We thus obtain

$$\xi(\tau) = \gamma^{-1}(\tau)\, \sigma(\tau) - \int_0^\tau \frac{d}{dt}\gamma^{-1}(t)\, \sigma(t)\, dt \tag{4.9}$$

The solution Function $y(t)$ is determined from this by the equation

$$y(t) = \beta^{-1}(t)\, h'(T, t)\, \xi(t) \tag{4.10}$$

In order to find the equations for the Lagrange multipliers in a closed form, we substitute the found $y(t)$ into the Expression (2.4) for the functional l_i. If we denote the ith column of the matrix $h(T, t)$ by $h_i(T, t)$, and introduce the matrices

$$\Gamma_i(t) = \int_t^T \frac{1}{\lambda_i(\tau)} h_i(T, \tau) h_i'(T, \tau)\, d\tau \qquad (i = 1, \ldots, m) \tag{4.11}$$

we obtain

$$l_i = -\frac{1}{\mu_i^2} \operatorname{sp} \int_0^T \dot{\Gamma}_i(t)\, \psi_{\xi\xi}(t)\, dt \tag{4.12}$$

In what follows we shall assume that $\dot{\psi}_{\sigma\sigma}(t)$ exists. Since by the definition of $\sigma(t)$

$$\psi_{\sigma\sigma}(t, \tau) = \psi_{\sigma\sigma}(\min\{t, \tau\})$$

the Equation (4.9) yields

$$\psi_{\xi\xi}(0) = \gamma^{-1}(0)\psi_{\sigma\sigma}(0)\gamma^{-1}(0), \qquad \dot{\psi}_{\xi\xi}(\tau) = \gamma^{-1}(\tau)\, \dot{\psi}_{\sigma\sigma}(\tau)\, \gamma^{-1}(\tau) \tag{4.13}$$

Furthermore, one can easily verify that $\dot{\psi}_{\sigma\sigma}(\tau)$ is a positive definite matrix.

Integration by parts of the Expression (4.12) leads to

$$l_i = \frac{1}{\mu_i^2} \operatorname{sp} \left[\gamma^{-1}(0)\, \Gamma_i(0)\, \gamma^{-1}(0)\, \psi_{\sigma\sigma}(0) + \int_0^T \gamma^{-1}(\tau)\, \Gamma_i(\tau)\, \gamma^{-1}(\tau)\, \dot{\psi}_{\sigma\sigma}(\tau)\, d\tau\right] \tag{4.14}$$

Except for the factor $1/\mu_i^2$, the μ_i enter into the Equations (4.14) only through $\gamma(\tau)$ which is given by the relation

$$\gamma(\tau) = E + \sum_{i=1}^{m} \frac{1}{\mu_i} \Gamma_i(\tau) \tag{4.15}$$

From the Equations (4.14) and (4.15) one can obtain the equations $l_i = k_i$ which determine the μ_i. The optimum $\epsilon^2(T)$ is obtained in the form

$$\varepsilon^2(T) = \operatorname{sp}\{\psi_{ss}(T) - \psi_{\sigma\sigma}(T)\} + l_0 \tag{4.16}$$

where

$$l_0 = \operatorname{sp}\left[\gamma^{-1}(0)\, \psi_{\sigma\sigma}(0)\, \gamma^{-1}(0) + \int_0^T \gamma^{-1}(t)\, \dot{\psi}_{\sigma\sigma}(t)\, \gamma^{-1}(t)\, dt\right] \tag{4.17}$$

In the general case, at least one solution is obtained in the form of a limit when some of the μ_i tend to zero (compare Section 5). With the aid of the Equations (3.12) it is not difficult to verify that the ith coordinate of $y(t)$ has an infinite number of possible values if $l_i < k_i$. Hence, in the given problem, the only case of interest, except for the limiting case, is the case of nonzero μ_i treated in this section.

5. In order to establish the existence of μ_i satisfying the conditions (3.5), we consider the quantity l defined by the relation

$$l = \operatorname{sp}\left[\gamma^{-1}(0)\,\psi_{\sigma\sigma}(0) + \int_0^T \gamma^{-1}(t)\,\dot{\psi}_{\sigma\sigma}(t)\,dt\right] \tag{5.1}$$

If one compares this expression with the Equations (4.14), (4.15) and (4.17), one can obtain

$$l = l_0 + \sum_{i=1}^{m} \mu_i\, l_i, \qquad l_i = \frac{\partial l}{\partial \mu_i} \tag{5.2}$$

Taking into account the fact that l and l_i are not negative, one can prove that l is a continuous function of μ_i in the closed region $\mu_i \geqslant 0$. Therefore, the function

$$H(\mu_1, \ldots, \mu_m) = l - \sum_{i=1}^{m} k_i \mu_i \tag{5.3}$$

will take on at some point $\mu_{i0} \geqslant 0$ a maximum value $H_{\max}$. If all the $\mu_{i0} > 0$, then $l_i = k_i$, and the $y(t)$ found in the preceding sections will be the sought optimal value. Let us now assume that, without restricting the generality,

$$\mu_{i0} > 0 \quad (i = 1, \ldots, p), \qquad \mu_{i0} = 0 \quad (i = p+1, \ldots, m)$$

For arbitrary $\mu_{i1} > 0$, let us consider

$$\mu_i = \begin{cases} \mu_{i0} & (i \leqslant p) \\ \mu\mu_{i1} & (i \geqslant p+1) \end{cases} \tag{5.4}$$

when $\mu \to 0$. One can show that, when $\mu \to 0$, $y(t)$ tends to a value $y^{\circ}(t)$ which is a solution of the Equations (3.12) when $\mu_i = \mu_{i0}$ $(i = 1, \ldots, m)$.

For this solution, the functionals l_i° are the limits of the functionals l_i. When $i \leqslant p$, we have $l_i^{\circ} = k_i$, i.e. the conditions (3.5) are satisfied. The remaining l_i° $(i = p+1, \ldots, m)$ depend on the values μ_{i1}. The following relations, similar to those of (5.2), will hold:

$$l^1 = \sum_{i=p+1}^{m} \mu_{i1}\, l_i^{\circ}, \qquad l_i^{\circ} = \frac{\partial l^1}{\partial \mu_{i1}} \tag{5.5}$$

where

$$l^1 = \operatorname{sp}\left[B_1(0)\,\psi_{\sigma\sigma}(0) + \int_0^T B_1(t)\,\dot{\psi}_{\sigma\sigma}(t)\,dt\right] \tag{5.6}$$

The positive semidefinite matrix $B_1(t)$ is determined by means of

$$A(t) = E + \sum_{i=1}^{p} \frac{1}{\mu_{i0}} \Gamma_i(t), \qquad B(t) = \sum_{i=p+1}^{m} \frac{1}{\mu_{i1}} \Gamma_i(t) \tag{5.7}$$

and by some matrix $U(t)$, through the equations

$$B(t) A^{-1}(t) = B(t) A^{-1}(t) B(t) B_1(t), \qquad B_1(t) = U(t) B(t) A^{-1}(t) \tag{5.8}$$

From $H \leqslant H_{max}$ follows the inequality

$$l^1 \leqslant \sum_{i=p+1}^{m} \mu_{i1} k_i \tag{5.9}$$

If $m = p + 1$, we obtain $l_m^{\circ} \leqslant k_m$ directly from (5.5) and (5.9). In order to prove in the general case the existence of those μ_{i1} for which the inequalities $l_i^{\circ} \leqslant k_i$ $(i = p + 1, \ldots, m)$ are satisfied, one has to establish, again, that l' is continuous in some closed region $\mu_{i1} \geqslant 0$. Therefore, the function l' will take a maximum value at some point μ_{i2} of the region $\mu_{i1} \geqslant 0$, $\Sigma\mu_{i1}k_i = 1$. For those i for which $\mu_{i2} > 0$, one can prove directly that $l_i^{\circ} \leqslant k_i$.

If some of the $\mu_{i2} = 0$, one has to go to the limit again, and repeat all arguments. Since $\Sigma\mu_{i2}k_i = 1$, not all the μ_{i2} are equal to zero, and one can prove by induction that there exists a solution $y(t)$ that satisfies the conditions $l_i \leqslant k_i$ $(i = p + 1, \ldots, m)$.

From the relations (5.6) and (5.9) it can be seen that the system of equations $l_i = k_i$ has a solution $\mu_i > 0$ for arbitrary (finite) $k_i > 0$ if for arbitrary $\mu_{i0} > 0$ $(i = 1, \ldots, p)$, and for some $\mu_{i1} > 0$ $(i = p + 1, \ldots, m)$,

$$\operatorname{sp} \int_0^T B_1(t)\, \dot{\psi}_{\sigma\sigma}(t)\, dt = \infty \tag{5.10}$$

(Hereby one must, of course, take into account all $p = 0, \ldots, m - 1$ and all possible permutations of the indices i.)

If $h(T, t)$ is an analytic function of t (this is true, in particular, in the case of a stationary object), and if $\dot{\psi}_{\sigma\sigma}(t)$ is a continuous function of t when $t = T$, then the following assertion is true. If the inequality

$$h_i'(T_1 t)\, \dot{\psi}_{\sigma\sigma}(T) = 0 \tag{5.11}$$

is false for every column $h_i(T, t)$ of the matrix $h(T, t)$ and for every t, then there exists, for every k_i a uniquely defined optimal system with positive μ_i.

6. As an application of the general results, we consider the problem of the synthesis of an automatic control system. Let us suppose that we are given an object whose motion is described by a system of linear differential equations, and that the external disturbances are random processes related to white noise by means of differential equations. By menas of a change of coordinates, the equations can always be reduced to the form

$$\zeta^{\cdot}(t) = C_0(t)\,\zeta(t) + C_1(t)\,y(t) + C_2(t)\,u(t) \tag{6.1}$$

where $\zeta(t)$ is a column of the coordinates of the system, $u(t)$ is white noise, and the C_i are given matrices. For the control elements we take some linear combinations $x(t) = M\zeta(t)$ of the coordinates of the system. Suppose that we are measuring linear combinations $N\zeta(t)$ with errors that are characterized by a process $n(t)$.

Just as above, we set ourselves the problem of minimizing $\epsilon^2(T)$ under the conditions $l_i < k_i$, but we are looking for a dependence $y(t)$ on ρ, and on $z(t) = N\zeta(t) + n(t)$. In the given case we obtain a system of linear differential equations. We are given random quantities ρ, ζ_0 (initial values of the coordinates (t)), and random processes $u(t)$ and $n(t)$. Let us suppose that $u(t)$ and $n(t)$ represent white noise and that they are not correlated, not in ρ and not in ζ_0, i.e.

$$\begin{gathered}\psi_{\rho u}(t) = 0, \quad \psi_{\zeta_0 u}(t) = 0, \quad \psi_{\rho n}(t) = 0, \quad \psi_{\zeta_0 n}(t) = 0 \\ \psi_{uu}(t, \tau) = E\delta(t-\tau), \quad \psi_{nn}(t, \tau) = E\delta(t-\tau), \quad \psi_{nu}(t, \tau) = 0\end{gathered} \tag{6.2}$$

The hypotheses on $n(t)$ have the physical meaning that all the quantities $N\zeta(t)$ are measured independently of each other, with high-frequency errors that do not depend on the external reactions or on the state of the system.

With regard to the column ρ we assume, without loss of generality, that either $\rho = 0$, or its components are linearly independent and $\psi_{\rho\rho} = E$.

7. In order to reduce this problem to the problem treated in the preceding sections, we introduce the matrix of the fundamental solutions $R(t)$, given by the equations

$$\dot{R}(t) = G_0(t)\,R(t), \qquad R(0) = E \tag{7.1}$$

and set, for the sake of brevity,

$$D_1(t) = R^{-1}(t)\,G_1(t), \quad D_2(t) = R^{-1}(t)\,G_2(t), \quad M_1 = MR(T) \tag{7.2}$$

The Equation (6.1) can then be reduced to the form (2.1), whereby

$$s(t) = MR(t)\left[\zeta_0 + \int_0^t D_2(\tau)\; u\;(\tau)\, d\tau\right] \tag{7.3}$$

$$h(t, \tau) = -\, MR(t)\, D_1(\tau) \tag{7.4}$$

In order to determine $r(\tau)$ in the expression for $z(t)$, we set $y \equiv 0$. We thus obtain

$$r(t) = NR(t)\left[\zeta_0 + \int_0^\tau D_2(\tau)\; u\;(\tau)\, d\tau\right] + n(t) \tag{7.5}$$

With the aid of the Equation (4.1) it is not difficult to verify that in our case

$$\sigma(t) = M_1 P^{-1}(t)\left[\psi_{\zeta_0\rho}\rho + \int_0^t Q(\tau)\, R'(\tau)\, N'\; r(\tau)\; d\tau\right] \tag{7.6}$$

Hereby,

$$P(t) = Q(t)\, R'(t)\, N'NR(t), \qquad P(0) = E$$

$$Q(t) = P(t)\, D_2(t)\, D_2'(t), \qquad Q(0) = \psi_{\zeta_0\zeta_0} - \psi_{\zeta_0\rho}\psi'_{\zeta_0\rho} \tag{7.7}$$

The Equations (4.9) and (4.10) yield $y(t)$ expressed in terms of $\sigma(t)$. This expression in conjunction with (7.6) and the equation expressing $r(t)$ in terms of $z(t)$ and $y(t)$, yields the sought system of equations. From (6.1) and (7.5) we obtain

$$r(t) = z(t) - NR(t)\int_0^t D_1(\tau)\; y(\tau)\; d\tau \tag{7.8}$$

and after a number of transformations, we obtain, finally,

$$y(t) = \beta^{-1}(t) D_1'(t) M_1' \gamma^{-1}(t)\; M_1 \eta(t) \tag{7.9}$$

Here the auxiliary column $\eta(t)$ satisfies the differential equation

$$\begin{aligned} \eta(t) + [P^{-1}(t)\, \dot{P}(t) - D_1(t)\beta^{-1}(t)\, D_1'(t)\; M_1'\gamma^{-1}(t)\; M_1]\; \eta(t) = \\ = -\, P^{-1}(t)\;\; Q(t)\; R'(t)\, N'z(t) \end{aligned} \tag{7.10}$$

with the initial condition

$$\eta(0) = \psi_{\zeta_0\rho}\,\rho \tag{7.11}$$

The matrices $\psi_{\sigma\sigma}(0)$ and $\dot{\psi}_{\sigma\sigma}(t)$ are obtained with the aid of Equations (7.5) and (7.6) in the form

$$\psi_{\sigma\sigma}(0) = M_1\psi_{\zeta_0\rho}[M_1\psi_{\zeta_0\rho}]'$$

$$\psi_{\sigma\sigma}(t) = M_1P^{-1}(t)\,Q(t)\,R'(t)\,N'\,[M_1P^{-1}(t)\,Q(t)\,R'(t)\,N']' \quad (7.12)$$

In the particular case when the quantities $x(t)$ are linear combinations of the measured quantities $N\zeta(t)$ (i.e. when $N = UM$), the criterion (5.11) indicates physically that for zero $y_j(t)$ ($j \neq i$) the optimal (in the sense of the least $\epsilon^2(T)$) control can be selected independently of $z(t)$ (i.e. the measurements $z(t)$ will not make it possible to decrease $\epsilon^2(T)$ with the aid of only the ith regulating element).

8. In a concrete case the computing procedure is as follows. One has to be given the matrices $G_0(t)$, $G_1(t)$, $G_2(t)$, N, M, $\psi_{\zeta_0\zeta_0}$, $\dot{\psi}_{\zeta_0\rho}$, with the weight functions $\lambda_i(t)$, and with the numbers k_i and T.

First one has to compute $R(t)$ as a solution of the system of differential Equations (7.1). Next one determines $R^{-1}(t)$, $D_1(t)$, $D_2(t)$, M_1, $\Gamma_i(t)$ (by the use of the Equations (7.4) and (4.11), $P(t)$, $Q(t)$ as the solution of the system of differential Equations (7.7), $P^{-1}(t)$, $\psi_{\sigma\sigma}(0)$ and $\dot{\psi}_{\sigma\sigma}(t)$. With the aid of the Equations (4.14) and (4.15), one determines the numbers μ_i from the system of equations $l_i = k_i$. After this one can compute $\beta(t)$, $\gamma(t)$, $\gamma^{-1}(t)$, find the optimal value $\epsilon^2(T)$ with the Equations (4.16) and (4.17) and obtain the optimal regulator from the Formulas (7.9), (7.10) and (7.11).

In the case of a stationary object and stationary noises, i.e. when G_0, G_1, G_2, N, M, and λ_i do not depend on t, $R(t)$ is determined from a system with constant coefficients. The Equations (7.7) can be reduced to a system with constant coefficients.

$$[P(t)\,R^{-1}(t)]^{\cdot} = -[P(t)\,R^{-1}(t)]\,G_0 + [Q(t)\,R'(t)]\,N'N$$
$$[Q(t)\,R'(t)]^{\cdot} = [P(t)\,R^{-1}(t)]\,G_2G_2' + [Q(t)\,R'(t)]\,G_0' \quad (8.1)$$

so that in this case the coefficients of the equations of the optimal control are rational expressions in t and exponential functions of time.

The subject matter of the present work is close to certain investigations published earlier. The approach to the solution (Sections 3,4) are based on general methods developed in [1]. The problem of the optimalization at a given time moment was considered for the discrete case by Booton [2].

If one denotes by ϵ_2° the optimal value $\epsilon^2(T)$, then the problem is equivalent to the minimizing of the functional $\Sigma\mu_i l_i$ among all controls for which $\epsilon^2(T) \leqslant \epsilon_2^{\circ}$. In this formulation, the problem is similar to the ones considered, for example, in [3]. The essential difference is that we were looking for a control that would be optimal under given (definite or random) initial conditions. This permitted us to confine ourselves to the knowledge of the second moments of the random functions.

The solution of the problem with restrictions (definition $\sigma(t)$, Section 7) was given in [4].

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